

Null controllability of the 1D heat equation using flatness

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Abstract: We derive in a straightforward way the null controllability of a 1-D heat equation with boundary control. We use the so-called *flatness approach*, which consists in parameterizing the solution and the control by the derivatives of a “flat output”. This provides an explicit control law achieving the exact steering to zero. We also give accurate error estimates when the various series involved are replaced by their partial sums, which is paramount for an actual numerical scheme. Numerical experiments demonstrate the relevance of the approach.

Keywords: Partial differential equations, heat equation, boundary control, null controllability, path planning, flatness.

1. INTRODUCTION

The controllability of the heat equation was first considered in the 1-D case, Fattorini and Russell (1971); Jones Jr. (1977); Littman (1978)), and very precise results were obtained by the classical moment approach. Next using Carleman estimates and duality arguments the null controllability was proved in Fursikov and Imanuvilov (1996); Lebeau and Robbiano (1995) for any bounded domain in \mathbb{R}^N , any control time T , and any control region. This Carleman approach proves very efficient also with semilinear parabolic equations, Fursikov and Imanuvilov (1996).

By contrast the numerical control of the heat equation (or of parabolic equations) is in its early stage, see e.g. Münch and Zuazua (2010); Boyer et al. (2011); Micu and Zuazua (2011). A natural candidate for the control input is the control of minimal L^2 -norm, which may be obtained as a trace of the solution of the (backward) adjoint problem whose terminal state is the minimizer of a suitable quadratic cost. Unfortunately its computation is a hard task Micu and Zuazua (2011); indeed the terminal state of the adjoint problem associated with some regular initial state of the control problem may be highly irregular, which leads to severe troubles in the numerical computation of the control function.

All the above results rely on some observability inequalities for the adjoint system. A direct approach which does not involve the adjoint problem was proposed in Jones Jr. (1977); Littman (1978); Lin Guo and Littman (1995). In Jones Jr. (1977) a fundamental solution for the heat equation with compact support in time was introduced and used to prove null controllability. The results in Jones Jr. (1977); Rosier (2002) can be used to derive control results on a bounded interval with one boundary control in some Gevrey class. An extension of those results to the semilinear heat equation in 1D was obtained in Lin Guo

and Littman (1995) in a more explicit way through the resolution of an ill-posed problem with data of Gevrey order 2 in t .

In this paper we derive in a straightforward way the null controllability of the 1-D heat equation

$$\theta_t(t, x) - \theta_{xx}(t, x) = 0, \quad (t, x) \in (0, T) \times (0, 1) \quad (1)$$

$$\theta_x(t, 0) = 0, \quad t \in (0, T) \quad (2)$$

$$\theta_x(t, 1) = u(t), \quad t \in (0, T) \quad (3)$$

with initial condition

$$\theta(0, x) = \theta_0(x), \quad x \in (0, 1).$$

This system describes the dynamics of the temperature θ in an insulated metal rod where the control u is the heat flux at one end. More precisely given any final time $T > 0$ and any initial state $\theta_0 \in L^2(0, 1)$ we provide an explicit control input $u \in L^2(0, T)$ such that the state reached at time T is zero, i.e.

$$\theta(T, x) = 0, \quad x \in (0, 1).$$

We use the so-called *flatness approach*, Fliess et al. (1995), which consists in parameterizing the solution θ and the control u by the derivatives of a “flat output” y (section 2); this notion was initially introduced for finite-dimensional (nonlinear) systems, and later extended to in particular parabolic PDEs, Laroche et al. (2000); Lynch and Rudolph (2002); Meurer and Zeitz (2008); Meurer (2011). Choosing a suitable trajectory for this flat output y then yields an explicit series for a control achieving the exact steering to zero (section 3). This generalizes Laroche et al. (2000), where only approximate controllability was achieved through a similar construction. We then give accurate error estimates when the various series involved are replaced by their partial sums, which is paramount for an actual numerical scheme (section 4). Numerical experiments demonstrate the relevance of the approach (section 5).

In the sequel we will consider series with infinitely many derivatives of some functions. The notion of Gevrey order is a way of estimating the growth of these derivatives: we say that a function $y \in C^\infty([0, T])$ is *Gevrey of order $s \geq 0$* on $[0, T]$ if there exist positive constants M, R such that

$$|y^{(p)}(t)| \leq M \frac{p!^s}{R^p} \quad \forall t \in [0, T], \quad \forall p \geq 0.$$

More generally if $K \subset \mathbb{R}^N$ is a compact set and y is a function of class C^∞ on K (i.e. y is the restriction to K of a function of class C^∞ on some open neighbourhood Ω of K), we say y is *Gevrey of order s_1 in x_1 , s_2 in x_2, \dots, s_N in x_N* on K if there exist positive constants M, R_1, \dots, R_N such that

$$|\partial_{x_1}^{p_1} \partial_{x_2}^{p_2} \dots \partial_{x_N}^{p_N} y(x)| \leq M \frac{\prod_{i=1}^N (p_i!)^{s_i}}{\prod_{i=1}^N R_i^{p_i}}, \quad \forall x \in K, \quad \forall p \in \mathbb{N}^N.$$

By definition, a Gevrey function of order s is also of order r for $r \geq s$. Gevrey functions of order 1 are analytic (entire if $s < 1$). Gevrey functions of order $s > 1$ have a divergent Taylor expansion; the larger s , the “more divergent” the Taylor expansion. Important properties of analytic functions generalize to Gevrey functions of order $s > 1$: the scaling, addition, multiplication and derivation of Gevrey functions of order s is of order s , see Ramis (1978); Rudin (1987). But contrary to analytic functions, functions of order $s > 1$ may be constant on an open set without being constant everywhere. For example the “step function”

$$\phi_s(t) := \begin{cases} 1 & \text{if } t \leq 0 \\ 0 & \text{if } t \geq 1 \\ \frac{e^{-(1-t)^{-k}}}{e^{-(1-t)^{-k}} + e^{-t^{-k}}} & \text{if } t \in]0, 1[\end{cases}$$

where $k = (s-1)^{-1}$ is Gevrey of order s on $[0, 1]$ (and in fact on \mathbb{R}); notice $\phi_s(0) = 1$, $\phi_s(1) = 0$ and $\phi_s^{(i)}(0) = \phi_s^{(i)}(1) = 0$ for all $i \geq 1$.

In conjunction with growth estimates we will repeatedly use Stirling’s formula $n! \sim (n/e)^n \sqrt{2\pi n}$.

2. THE HEAT EQUATION IS FLAT

We claim the system (1)–(3) is “flat” with $y(t) := \theta(0, t)$ as a flat output, which means there is (in appropriate spaces of smooth functions) a 1–1 correspondence between arbitrary functions $t \mapsto y(t)$ and solutions of (1)–(3).

We first seek a formal solution in the form

$$\theta(t, x) := \sum_{i \geq 0} \frac{x^i}{i!} a_i(t)$$

where the a_i ’s are functions yet to define. Plugging this expression into (1) yields

$$\sum_{i \geq 0} \frac{x^i}{i!} [a_{i+2} - a_i'] = 0,$$

hence $a_{i+2} = a_i'$ for all $i \geq 0$. On the other hand $y(t) = \theta(0, t) = a_0(t)$, and (2) implies $a_1(t) = 0$. As a consequence $a_{2i} = y^{(i)}$ and $a_{2i+1} = 0$ for all $i \geq 0$. The formal solution thus reads

$$\theta(t, x) = \sum_{i \geq 0} \frac{x^{2i}}{(2i)!} y^{(i)}(t) \quad (4)$$

while the formal control is given by

$$u(t) = \theta_x(1, t) = \sum_{i \geq 1} \frac{y^{(i)}(t)}{(2i-1)!}. \quad (5)$$

We now give a meaning to this formal solution by restricting $t \mapsto y(t)$ to be Gevrey of order $s \in [0, 2)$.

Proposition 1. Let $s \in [0, 2)$, $-\infty < t_1 < t_2 < \infty$, and $y \in C^\infty([t_1, t_2])$ satisfying for some constants $M, R > 0$

$$|y^{(i)}(t)| \leq M \frac{i!^s}{R^i}, \quad \forall i \geq 0, \quad \forall t \in [t_1, t_2]. \quad (6)$$

Then the function θ defined by (4) is Gevrey of order s in t and $s/2$ in x on $[t_1, t_2] \times [0, 1]$; hence the control u defined by (5) is also Gevrey of order s on $[t_1, t_2]$.

Proof. We must prove the formal series

$$\partial_t^m \partial_x^n \theta(t, x) = \sum_{2i \geq n} \frac{x^{2i-n}}{(2i-n)!} y^{(i+n)}(t) \quad (7)$$

is uniformly convergent on $[t_1, t_2] \times [0, 1]$ with growth estimates of the form

$$|\partial_t^m \partial_x^n \theta(t, x)| \leq C \frac{m!^s}{R_1^m} \frac{n!^{\frac{s}{2}}}{R_2^n}. \quad (8)$$

By (6), we have for all $(t, x) \in [t_1, t_2] \times [0, 1]$

$$\begin{aligned} \left| \frac{x^{2i-n}}{(2i-n)!} y^{(i+m)}(t) \right| &\leq \frac{M}{R^{i+m}} \frac{(i+m)!^s}{(2i-n)!} \\ &\leq \frac{M}{R^{i+m}} \frac{(2^{i+m} i! m!)^s}{(2i-n)!} \\ &\leq \frac{M}{R^{i+m}} \frac{2^{si} (2^{-2i} \sqrt{\pi i} (2i)!)^{\frac{s}{2}} m!^s}{(2i-n)!} \frac{m!^s}{2^{-sm}} \\ &\leq M \frac{(\pi i)^{\frac{s}{4}}}{R_1^i (2i-n)!^{1-\frac{s}{2}}} \frac{n!^{\frac{s}{2}} m!^s}{R_1^m}, \end{aligned}$$

where we have set $R_1 = 2^{-s} R$; we have used Stirling’s formula for $(2i)!$ and twice $(i+j)! \leq 2^{i+j} i! j!$. Since $\sum_{2i \geq n} \frac{(\pi i)^{\frac{s}{4}}}{R_1^i (2i-n)!^{1-\frac{s}{2}}} < \infty$ the series in (7) are uniformly convergent for all $m, n \geq 0$, hence $\theta \in C^\infty([t_1, t_2] \times [0, 1])$. Finally, since

$$\sum_{2i \geq n} \frac{M(\pi i)^{\frac{s}{4}}}{R_1^i (2i-n)!^{1-\frac{s}{2}}} \leq M \left(\frac{\pi}{2} \right)^{\frac{s}{4}} R_1^{-\frac{n}{2}} \sum_{j \geq 0} \frac{j^{\frac{s}{4}} + n^{\frac{s}{4}}}{R_1^{\frac{j}{2}} j!^{1-\frac{s}{2}}} \leq C R_2^{-n}$$

where $R_2 \in (0, \sqrt{R_1})$ and $C > 0$ is some constant independent of n , we have the desired estimates (8). \square

3. NULL CONTROLLABILITY

In this section we derive an explicit control steering the system from any initial state $\theta_0 \in L^2(0, 1)$ at time 0 to the final state 0 at time $T > 0$. Two ideas are involved: on the one hand thanks to the flatness property it is easy to find a control achieving the steering to zero starting from a certain set of initial conditions (lemma 2); on the other hand thanks to the regularizing property of the heat equation this set is reached from any $\theta_0 \in L^2(0, 1)$ when applying first a zero control for some time (lemma 3).

Lemma 2. Let $(y_i)_{i \geq 0}$ be a sequence of real numbers such that for some constants $M, R > 0$

$$|y_i| \leq M \frac{i!}{R^i} \quad \forall i \geq 0. \quad (9)$$

Then the function defined on $[t_1, t_1 + R']$, $R' < R$, by

$$y(t) := \phi_s\left(\frac{t-t_1}{R'}\right) \sum_{i \geq 0} y_i \frac{(t-t_1)^i}{i!},$$

is Gevrey of order $s > 1$ on $[t_1, t_1 + R']$ and satisfies for all $i \geq 0$

$$y^{(i)}(t_1) = y_i \quad (10)$$

$$y^{(i)}(t_1 + R') = 0. \quad (11)$$

Moreover the control defined on $[t_1, t_1 + R']$ by

$$u(t) := \sum_{i \geq 1} \frac{y^{(i)}(t)}{(2i-1)!} \quad (12)$$

is also Gevrey of order s on $[t_1, t_1 + R']$ and steers the system from the initial state $\sum_{i \geq 0} y_i \frac{x^{2i}}{(2i)!}$ at time t_1 to the final state 0 at time $t_1 + R'$.

Proof. Let $\bar{y}(z) := \sum_{i \geq 0} y_i \frac{z^i}{i!}$. The growth property (9) implies \bar{y} is analytic on the disc $\{z \in \mathbb{C}; |z| < R\}$, the convergence being moreover uniform for $|z| \leq R' < R$. Therefore \bar{y} is Gevrey of order 1, hence of order $s > 1$, on $[t_1, t_1 + R']$. On the other hand $\phi(t) := \phi_s\left(\frac{t-t_1}{R'}\right)$ is also Gevrey of order s on $[t_1, t_1 + R']$, hence so is the product y of \bar{y} and ϕ . The boundary values (10)-(11) follow at once from the definition of ϕ_s .

The control u in (12) achieves the steering to zero; indeed

$$\theta(t, x) := \sum_{i \geq 0} \frac{x^{2i}}{(2i)!} y^{(i)}(t),$$

as well as u , is by proposition 1 Gevrey of order s in t and $s/2$ in x , and obviously satisfies $\theta(t_1 + R', x) = 0$. \square

Lemma 3. Let $\theta_0 \in L^2(0, 1)$ and $\tau > 0$. Consider the final state $\theta_\tau(x) := \theta(\tau, x)$ reached when applying the control $u(t) := 0$, $t \in [0, \tau]$, starting from the initial state θ_0 .

Then θ_τ is analytic in \mathbb{C} and can be expanded as

$$\theta_\tau(x) = \sum_{i \geq 0} y_i \frac{x^{2i}}{(2i)!}, \quad x \in \mathbb{C},$$

with

$$|y_i| \leq C \left(1 + \frac{1}{\sqrt{\tau}}\right) \frac{i!}{\tau^i}$$

where C is some positive constant depending only on θ_0 .

Proof. Decompose θ_0 as the Fourier series of cosines

$$\theta_0(x) = \sum_{n \geq 0} c_n \sqrt{2} \cos(n\pi x)$$

where the convergence holds in $L^2(0, 1)$ and

$$2|c_0|^2 + \sum_{n \geq 1} |c_n|^2 = \int_0^1 |\theta_0(x)|^2 dx < \infty.$$

The solution starting from θ_0 then reads

$$\theta(t, x) = \sum_{n \geq 0} c_n e^{-n^2 \pi^2 t} \sqrt{2} \cos(n\pi x) \quad (13)$$

and in particular

$$\theta_\tau(x) = \sum_{n \geq 0} c_n e^{-n^2 \pi^2 \tau} \sqrt{2} \cos(n\pi x).$$

The series for θ_τ is analytic in \mathbb{C} since for all $|x| \leq r$

$$\left| c_n e^{-n^2 \pi^2 \tau} \sqrt{2} \cos(n\pi x) \right| \leq C_1 \left(\sup_{k \geq 0} |c_k| \right) e^{-n^2 \pi^2 \tau + n\pi r}$$

where C_1 is some positive constant; this ensures the uniform convergence of the series in every open disk of radius $r > 0$.

Moreover

$$\begin{aligned} \theta_\tau(x) &= \sqrt{2} \sum_{n \geq 0} c_n e^{-n^2 \pi^2 \tau} \sum_{i \geq 0} (-1)^i \frac{(n\pi x)^{2i}}{(2i)!} \\ &= \sum_{i \geq 0} \frac{x^{2i}}{(2i)!} \underbrace{\left(\sqrt{2} (-1)^i \sum_{n \geq 0} c_n e^{-n^2 \pi^2 \tau} (n\pi)^{2i} \right)}_{=: y_i} \end{aligned}$$

The change in the order of summation will be justified once we have proved that y_i , $i \geq 0$, is absolutely convergent and

$$\sum_{i \geq 0} |y_i| \frac{x^{2i}}{(2i)!} < \infty, \quad \forall x \geq 0.$$

For $i \geq 0$ let $h_i(x) := e^{-\tau \pi^2 x^2} (\pi x)^{2i}$ and $N_i := \left\lceil \left(\frac{i}{\pi^2 \tau} \right)^{\frac{1}{2}} \right\rceil$.

The map h_i is increasing on $[0, \left(\frac{i}{\pi^2 \tau} \right)^{\frac{1}{2}}]$ and decreasing on $\left[\left(\frac{i}{\pi^2 \tau} \right)^{\frac{1}{2}}, +\infty \right)$ hence

$$\begin{aligned} \sum_{n \geq 0} h_i(n) &\leq \int_0^{N_i} h_i(x) dx + h_i(N_i) \\ &\quad + h_i(N_i + 1) + \int_{N_i+1}^{\infty} h_i(x) dx \\ &\leq 2h_i\left(\left(\frac{i}{\pi^2 \tau}\right)^{\frac{1}{2}}\right) + \int_0^{\infty} h_i(x) dx \\ &\leq C_2 \frac{i!}{\tau^i \sqrt{i\tau}} + \int_0^{\infty} h_i(x) dx; \end{aligned}$$

C_2 is some positive constant and we have used Stirling's formula. On the other hand integrating by parts yields

$$\begin{aligned} \int_0^{\infty} h_i(x) dx &= \frac{2i-1}{2\tau} \int_0^{\infty} h_{i-1}(x) dx \\ &= \frac{(2i-1) \cdots 3 \cdot 1}{(2\tau)^i} \int_0^{\infty} e^{-\tau \pi^2 x^2} dx \\ &= \frac{(2i)!}{2^i i! (2\tau)^i} \cdot \frac{1}{\pi \sqrt{\tau}} \int_0^{\infty} e^{-x^2} dx \\ &\leq C_3 \frac{i!}{\tau^i \sqrt{i\tau}}, \end{aligned}$$

where C_3 is some positive constant and we have again used Stirling's formula. As a consequence

$$|y_i| \leq \sqrt{2} \sup_{n \geq 0} |c_n| \sum_{n \geq 0} h_i(n) \leq C \left(1 + \frac{1}{\sqrt{\tau}}\right) \frac{i!}{\tau^i} \quad (14)$$

where C is some positive constant. Finally

$$\sum_{i \geq 0} |y_i| \frac{x^{2i}}{(2i)!} \leq C \left(1 + \frac{1}{\sqrt{\tau}}\right) \sum_{i \geq 0} \underbrace{\frac{i!}{(2i)!} \left(\frac{x^2}{\tau}\right)^i}_{=: v_i} < \infty$$

since $\frac{v_{i+1}}{v_i} \sim \frac{1}{4i} \frac{x^2}{\tau}$. \square

With the two previous lemma at hand we can now state our main controllability result.

Theorem 4. Let $\theta_0 \in L^2(0, 1)$ and $T > 0$. Pick any $\tau \in (0, T)$ and $s \in (1, 2)$. Then there exists a function y Gevrey of order s on $[\tau, T]$ such that the control

$$u(t) := \begin{cases} 0 & \text{if } 0 \leq t \leq \tau \\ \sum_{i \geq 1} \frac{y^{(i)}(t)}{(2i-1)!} & \text{if } \tau < t \leq T. \end{cases}$$

steers the system from the initial state θ_0 at time 0 to the final state 0 at time T .

Moreover u is Gevrey of order s on $[0, T]$; $t \mapsto \theta(t, \cdot)$ is in $C([0, T], L^2(0, 1))$; θ is Gevrey of order s in t and $s/2$ in x on $[\varepsilon, T] \times [0, 1]$ for all $\varepsilon \in (0, T)$.

Proof. By lemma 3 the state reached at time τ reads $\sum_{i \geq 0} y_i \frac{x^{2i}}{(2i)!}$ with the sequence $(y_i)_{i \geq 0}$ satisfying the growth property of lemma 2 with $M := 1 + \frac{1}{\sqrt{\tau}}$ and $R := \tau$. Hence the desired function is given by

$$y(t) := \phi_s\left(\frac{t-\tau}{R'}\right) \sum_{i \geq 0} y_i \frac{(t-\tau)^i}{i!},$$

on $[\tau, \tau + R']$, where $R' < \tau$ and $R' \leq T - \tau$; and by $y(t) := 0$ on $[\tau + R', T]$. Moreover y and u are Gevrey of order s on $[\tau, T]$, and by construction $\theta(\tau, x) = \theta(\tau^+, x) = \sum_{i \geq 0} y_i \frac{x^{2i}}{(2i)!}$ and $u(\tau) = u(\tau^+) = 0$. By Proposition 1 the solution θ on $[\tau, T] \times [0, 1]$ is well-defined and Gevrey of order s in t and $s/2$ in x hence $t \mapsto \theta(t, \cdot) \in C([0, T]; C^1([0, 1]))$ and $u \in C([0, T])$. On the other hand it is easily seen that θ is Gevrey of order 1 in t and $1/2$ in x on $[\varepsilon, \tau] \times [0, 1]$ for all $\varepsilon \in (0, \tau)$.

Thus the solution θ is Gevrey of order 1 in t and $1/2$ in x on $[\varepsilon, \tau] \times [0, 1]$ while it is Gevrey of order s in t and $s/2$ in x on $[\tau, T] \times [0, 1]$. To prove θ is Gevrey of order s in t and $s/2$ in x on $[\varepsilon, T] \times [0, 1]$ it is then sufficient to check $\partial_t^k \theta(\tau, x) = \partial_t^k \theta(\tau^+, x)$ for $k \geq 0$ and $x \in [0, 1]$. But

$$\begin{aligned} \partial_t^k \theta(\tau^+, x) &= \sum_{i \geq 0} \frac{x^{2i}}{(2i)!} y^{(i+k)}(\tau) \\ &= \sum_{i \geq 0} \frac{x^{2i}}{(2i)!} y_{i+k} \\ &= \sqrt{2} \sum_{i \geq 0} \frac{x^{2i}}{(2i)!} \left(\sum_{n \geq 0} c_n e^{-n^2 \pi^2 \tau} n^{2(i+k)} \right) (-\pi^2)^{i+k} \\ &= \sum_{n \geq 0} c_n (-n^2 \pi^2)^k e^{-n^2 \pi^2 \tau} \sqrt{2} \cos(n\pi x) \\ &= \partial_t^k \theta(\tau, x). \end{aligned}$$

As a consequence u is also Gevrey of order s on $[0, T]$. \square

4. NUMERICAL ESTIMATES

Summarizing the previous section the control u and solution θ on $[\tau, \tau + R']$ are given by the infinite series

$$u(t) = \sum_{i \geq 1} \frac{y^{(i)}(t)}{(2i-1)!} \quad (15)$$

$$\theta(t, x) = \sum_{i \geq 0} y^{(i)}(t) \frac{x^{2i}}{(2i)!} \quad (16)$$

$$y(t) = \phi_s\left(\frac{t-\tau}{R'}\right) \sum_{k \geq 0} y_k \frac{(t-\tau)^k}{k!} \quad (17)$$

$$y_k = \sqrt{2} \left(\sum_{n \geq 0} c_n e^{-n^2 \pi^2 \tau} n^{2k} \right) (-\pi^2)^k; \quad (18)$$

moreover y hence u and θ are identically zero on $[\tau + R', T]$. The aim of this section is to show that the partial sums

$$\bar{u}(t) := \sum_{1 \leq i \leq \bar{i}} \frac{y^{(i)}(t)}{(2i-1)!} \quad (19)$$

$$\bar{\theta}(t, x) := \sum_{0 \leq i \leq \bar{i}} y^{(i)}(t) \frac{x^{2i}}{(2i)!} \quad (20)$$

$$\bar{y}(t) := \phi_s\left(\frac{t-\tau}{R'}\right) \sum_{0 \leq k \leq \bar{k}} y_k \frac{(t-\tau)^k}{k!} \quad (21)$$

$$\bar{y}_k := \sqrt{2} \left(\sum_{0 \leq n \leq \bar{n}} c_n e^{-n^2 \pi^2 \tau} n^{2k} \right) (-\pi^2)^k. \quad (22)$$

for given $\bar{i}, \bar{k}, \bar{n} \in \mathbb{N}$ provide very good approximations of the above series, and to give explicit error estimates.

Theorem 5. There exist positive constants C, C_1, C_2, C_3 such that for all $\theta_0 \in L^2$, $\bar{i}, \bar{k}, \bar{n} \in \mathbb{N}$, and $t \in [\tau, T]$

$$\|\theta(t) - \bar{\theta}(t)\|_{L^\infty} \leq C \left(e^{-C_1 \bar{i} \ln \bar{i}} + e^{-C_2 \bar{k}} + e^{-C_3 \bar{n}^2} \right) \|\theta_0\|_{L^2}$$

Proof. First notice that for $(t, x) \in [\tau, T] \times [0, 1]$

$$|\theta(t, x) - \bar{\theta}(t, x)| \leq \Delta_1 + \Delta_2 + \Delta_3,$$

where

$$\begin{aligned} \Delta_1 &:= \left| \sum_{i > \bar{i}} y^{(i)}(t) \frac{x^{2i}}{(2i)!} \right| \\ \Delta_2 &:= \left| \sum_{0 \leq i \leq \bar{i}} \partial_t^i \left[\phi(t) \sum_{k > \bar{k}} y_k \frac{(t-\tau)^k}{k!} \right] \frac{x^{2i}}{(2i)!} \right| \\ \Delta_3 &:= \left| \sum_{0 \leq i \leq \bar{i}} \partial_t^i \left[\phi(t) \sum_{0 \leq k \leq \bar{k}} y_k \frac{(t-\tau)^k}{k!} \right] \frac{x^{2i}}{(2i)!} \right| \\ &\quad \sqrt{2} \left(\sum_{n > \bar{n}} c_n e^{-n^2 \pi^2 \tau} n^{2k} \right) (-\pi^2)^k \frac{(t-\tau)^k}{k!} \left| \frac{x^{2i}}{(2i)!} \right|. \end{aligned}$$

By lemma 2 y is Gevrey of order s on $[\tau, T]$ with some $M_1 \|\theta_0\|_{L^2}$, $R_1 > 0$ hence

$$\begin{aligned} \Delta_1 &\leq \sum_{i > \bar{i}} \frac{|y^{(i)}(t)|}{(2i)!} \\ &\leq M_1 \|\theta_0\|_{L^2} \sum_{i > \bar{i}} \frac{i!^s}{(2i)! R_1^i} \\ &\leq M'_1 \|\theta_0\|_{L^2} \sum_{i > \bar{i}} \frac{\sqrt{i}}{(4R_1)^i i^{\sqrt{i}^{2-s}}} \left(\frac{i}{e} \right)^{(s-2)i} \\ &\leq M'_1 \|\theta_0\|_{L^2} \sum_{i > \bar{i}} \frac{i^{\frac{s-1}{2}} e^{(2-s)(1-\ln i)i}}{(4R_1)^i}, \end{aligned}$$

where we have used Stirling's formula. Pick $C_1 < 2 - s$ and $\sigma \in (C_1, 2 - s)$. Then for $i > \bar{i}$

$$(4R_1)^{-i} i^{\frac{s-1}{2}} e^{(2-s)(1-\ln i)i} \leq K_1 e^{-\sigma i(\ln i-1)},$$

where $K_1 = K_1(s, \sigma, R_1)$. But

$$\begin{aligned} \sum_{i > \bar{i}} e^{-\sigma i(\ln i-1)} &\leq \int_{\bar{i}}^{\infty} e^{-\sigma x(\ln x-1)} dx \\ &\leq K'_1 \int_{\bar{i}(\ln \bar{i}-1)}^{\infty} e^{-\sigma x} dx \\ &\leq K'_1 e^{-C_1 \bar{i} \ln \bar{i}}, \end{aligned}$$

so that eventually

$$\Delta_1 \leq K''_1 \|\theta_0\|_{L^2} e^{-C_1 \bar{i} \ln \bar{i}}.$$

For Δ_2 we first notice that for $t \in \{z \in \mathbb{C}; |z - \tau| \leq \rho'\tau\}$, where ρ' satisfies $R'/\tau < \rho' < 1$,

$$\begin{aligned} \left| \sum_{k > \bar{k}} y_k \frac{(z - \tau)^k}{k!} \right| &\leq \sum_{k > \bar{k}} |y_k| \frac{\rho'^k \tau^k}{k!} \\ &\leq M_1 \|\theta_0\|_{L^2} \sum_{k > \bar{k}} \rho'^k \\ &\leq M''_1 \|\theta_0\|_{L^2} \rho'^{\bar{k}+1}. \end{aligned}$$

By the Cauchy estimates we thus have for $\tau \leq t \leq \tau + R'$

$$\left| \partial_t^i \left[\sum_{k > \bar{k}} y_{j,k} \frac{(t - \tau)^k}{k!} \right] \right| \leq M''_1 \|\theta_0\|_{L^2} \rho'^{\bar{k}+1} \frac{i!}{R_1^i},$$

hence

$$\left| \partial_t^i [\phi(t) \sum_{k > \bar{k}} y_k \frac{(t - \tau)^k}{k!}] \right| \leq M_2 \|\theta_0\|_{L^2} \rho'^{\bar{k}+1} \frac{i!^s}{R_2^i}.$$

It follows that

$$\Delta_2 \leq \sum_{0 \leq i \leq \bar{i}} M_2 \|\theta_0\|_{L^2} \rho'^{\bar{k}+1} \frac{i!^s}{(2i)! R_3^i} \leq K_2 \|\theta_0\|_{L^2} e^{-C_2 \bar{k}}$$

where $0 < C_2 < \ln \frac{\tau}{R'}$.

To estimate Δ_3 first notice that for $\alpha > 0$

$$\sum_{n > \bar{n}} e^{-\alpha n^2} \leq \int_{\bar{n}}^{\infty} e^{-\alpha x^2} dx = \int_{\bar{n}}^{\infty} \frac{e^{-\alpha y}}{2\sqrt{y}} dy \leq \frac{e^{-\alpha \bar{n}^2}}{2\alpha \bar{n}}.$$

Pick ρ' and ρ'' with $R\tau < \rho' < \rho'' < 1$. Then

$$\begin{aligned} \left| \sum_{n > \bar{n}} c_n e^{-n^2 \pi^2 \tau} n^{2k} \right| &\leq \|\theta_0\|_{L^2} \frac{e^{-k} k^k}{(\pi^2 \rho'' \tau)^k} \sum_{n > \bar{n}} e^{-n^2 \pi^2 (1-\rho'')\tau} \\ &\leq K_3 \|\theta_0\|_{L^2} \frac{k!}{(\pi^2 \rho'' \tau)^k} \frac{e^{-\pi^2 (1-\rho'')\tau \bar{n}^2}}{\bar{n}}, \end{aligned}$$

where we have used first that $x \rightarrow e^{-x} x^k$ is maximum for $x = k$, and then that $e^{-k} k^k \leq Ck!$ by Stirling's formula. Hence for $z \in \{z \in \mathbb{C}; |z - \tau| \leq \rho'\tau\}$

$$\begin{aligned} \left| \sum_{0 \leq k \leq \bar{k}} \sqrt{2} \left(\sum_{n > \bar{n}} c_{j,n} e^{-n^2 \pi^2 \tau} n^{2k} \right) (-\pi^2)^k \frac{(z - \tau)^k}{k!} \right| \\ \leq K'_3 \|\theta_0\|_{L^2} \sum_{0 \leq k \leq \bar{k}} \frac{e^{-\pi^2 (1-\rho'')\tau \bar{n}^2}}{\bar{n}} \left(\frac{\rho'}{\rho''} \right)^k \\ \leq K''_3 \|\theta_0\|_{L^2} \frac{e^{-\pi^2 (1-\rho'')\tau \bar{n}^2}}{\bar{n}}. \end{aligned}$$

Setting $C_3 := \pi^2(1 - \rho'')\tau < \pi^2(\tau - R')$, this yields for $\tau \leq t \leq \tau + R'$

$$\begin{aligned} \left| \partial_t^i \left[\sum_{0 \leq k \leq \bar{k}} \sqrt{2} \left(\sum_{n > \bar{n}} c_n e^{-n^2 \pi^2 \tau} n^{2k} \right) (-\pi^2)^k \frac{(t - \tau)^k}{k!} \right] \right| \\ \leq M_3 \|\theta_0\|_{L^2} e^{-C_3 \bar{n}^2} \frac{i!}{R_1^i} \end{aligned}$$

and

$$\begin{aligned} \left| \partial_t^i [\phi(t) \sum_{0 \leq k \leq \bar{k}} \sqrt{2} \left(\sum_{n > \bar{n}} c_{j,n} e^{-n^2 \pi^2 \tau} n^{2k} \right) (-\pi^2)^k \frac{(t - \tau)^k}{k!}] \right| \\ \leq M'_3 \|\theta_0\|_{L^2} e^{-C_3 \bar{n}^2} \frac{i!^s}{R_2^i}. \end{aligned}$$

We then conclude

$$\Delta_3 \leq \sum_{0 \leq i \leq \bar{i}} M'_3 \|\theta_0\|_{L^2} e^{-C_3 \bar{n}^2} \frac{i!^s}{(2i)! R_2^i} \leq M''_3 \|\theta_0\|_{L^2} e^{-C_3 \bar{n}^2}$$

Collecting the inequalities for $\Delta_1, \Delta_2, \Delta_3$ eventually gives the statement of the theorem. \square

Let $\hat{\theta}$ denote the solution of (1)–(3) where the “exact” control u (15) is replaced by the truncated control \bar{u} (19), still starting from the initial condition θ_0 . Notice $\hat{\theta}$ is the “exact” solution obtained when applying the truncated control, while θ is the truncated solution obtained when applying the “exact” control. Since in “real life” only the truncated control can be actually applied it is important to know how well $\hat{\theta}$ approximates θ , the “exact” solution when applying the “exact” control. It turns that $\hat{\theta}$ satisfies the same relation as $\bar{\theta}$ in Theorem 5; the proof is omitted for lack of space but follows from the proof of Theorem 5.

Corollary 6. With the same notations as in Theorem 5,

$$\|\theta(t) - \hat{\theta}(t)\|_{L^\infty} \leq C \left(e^{-C_1 \bar{i} \ln \bar{i}} + e^{-C_2 \bar{k}} + e^{-C_3 \bar{n}^2} \right) \|\theta_0\|_{L^2}$$

5. NUMERICAL EXPERIMENTS

We have conducted a number of numerical experiments to demonstrate the relevance of our approach, and to investigate the influence of the main parameters. We have focused on the control effort, which is probably the most important quantity. The results are summarized in the two tables below, where $\|\bar{u}\|_{L^2}$ (top table) and $\|\bar{u}\|_{L^\infty}$ (bottom table) are given for different values of the parameters s and R' in (21). Figures 1 and 2 give moreover the complete temperature and control evolution for the case $R' = 0.2$ and $s = 1.6$. For all experiments the “regularization time” τ is 0.3, and the initial condition θ_0 is a step function with $\theta_0(x) = -1$ on $[0, 1/2)$ and $\theta_0(x) = 1$ on $(1/2, 1)$ hence Fourier coefficients $c_{2p} = 0$ and $c_{2p+1} = \frac{(-1)^{p+1}}{2p+1} \frac{2\sqrt{2}}{\pi}$ for $p \geq 0$; $\bar{i}, \bar{k}, \bar{n}$ are “large enough” for a good accuracy of the truncated series (19)–(22).

s, R'	0.15	0.20	0.25	0.30
1.5	693	63.3	12.7	3.82
1.6	35.3	6.41	1.95	0.78
1.7	7.49	1.95	0.74	0.34
1.8	5.53	1.24	0.48	0.23
1.9	5.71	1.29	0.47	0.22

s, R'	0.15	0.20	0.25	0.30
1.5	3666	330	55.2	18.1
1.6	118	23.6	7.17	2.76
1.7	18.6	4.78	1.73	0.76
1.8	36.4	4.59	1.44	0.65
1.9	47.8	9.66	2.13	0.82

A few conclusions can be drawn from these experiments:

- when R' gets small there seems to be an “optimal” value of s which minimizes the control effort (the values are different for $\|\bar{u}\|_{L^2}$ and $\|\bar{u}\|_{L^\infty}$)
- when s is too small or too large the derivatives $\bar{y}^{(i)}$ tend to “crowd” near the extremities of $[\tau, \tau + R']$ with large amplitudes
- the smaller R' the more noticeable this phenomenon, with of course an increased control effort.

An interesting question is the tradeoff between τ and R' to reach the zero state at time $T := \tau + R'$ with the smallest control effort: longer regularization τ or longer duration R' of the active control? With the present construction of the function y (18) we have by design $R' \leq \tau$, which is probably an important restriction. Other, but more complicated, constructions without this limitation are possible and will be studied in the future.

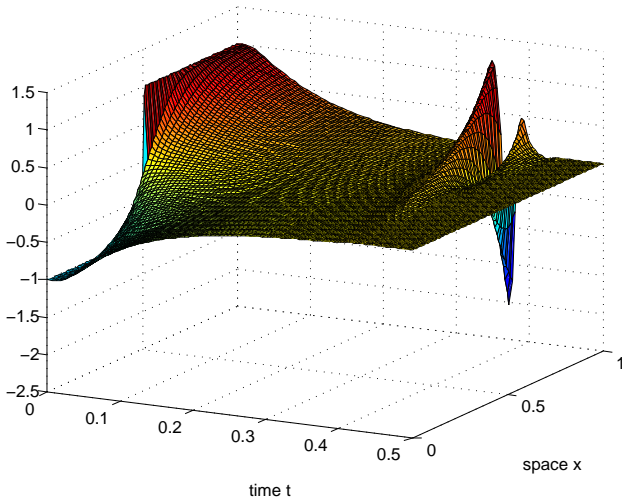


Fig. 1. $\bar{\theta}(t, x)$ for $R' = 0.2$ and $s = 1.6$.

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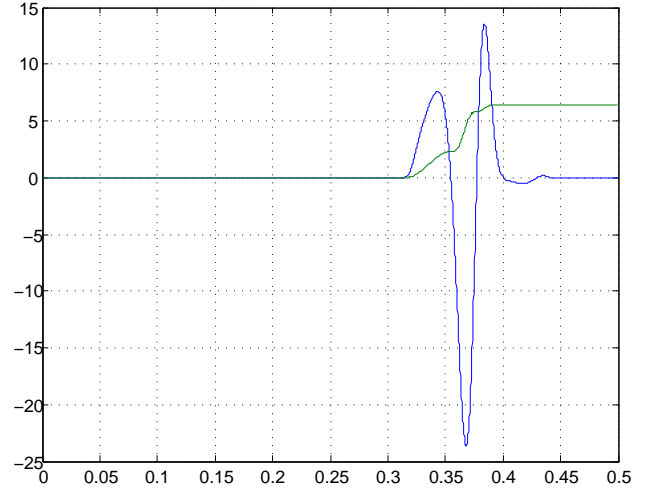


Fig. 2. $\bar{u}(t)$ and $\|\bar{u}\|_{L^2(0,t)}$ for $R' = 0.2$ and $s = 1.6$.

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